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ADDENDUM TO “THE CONVEX-HULL LIKE PROPERTY AND SUPPORTED IMAGES OF OPEN SETS”

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A function $\psi : \mathbf{R}^n \to \mathbf{R}$ is said to be quasi-convex if, for each $r \in \mathbf{R}$, the set $\psi^{-1}(]-\infty, r])$ is convex.

In the sequel, $\Omega \subset \mathbf{R}^n$ is a non-empty open bounded set.

Let $f : \Omega \to \mathbf{R}^n$ be a continuous function.

In [2], we introduced the following definition: the function f is said to satisfy the convex hull-like property if, for every continuous and quasi-convex function $\psi : \mathbf{R}^n \to \mathbf{R}$, there exists $x^* \in \partial\Omega$ such that

$$\limsup_{x \to x^*} \psi(f(x)) = \sup_{x \in \Omega} \psi(f(x)) .$$

In [2], we also remarked that, given a continuous function $g : \overline{\Omega} \to \mathbf{R}^n$, the function $g_{|\Omega}$ satisfies the convex hull-like property if and only if

$$g(\Omega) \subseteq \mathrm{conv}(g(\partial\Omega)) ,$$

$\mathrm{conv}(g(\partial\Omega))$ being the convex hull of $g(\partial\Omega)$.

Further, we recall that a set $S \subseteq \mathbf{R}^n$ is said to be supported at the point $y_0 \in S$ if there exists a non-zero linear function $\varphi : \mathbf{R}^n \to \mathbf{R}$ such that $\varphi(y_0) \leq \varphi(y)$ for all $y \in S$.

If this happens, of course $y_0 \in \partial S$.

The basic result of [2] is as follows:

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Theorem 0.1 ([2], Theorem 1.5)**.** *For any continuous function $f : \Omega \to \mathbf{R}^n$, at least one of the following assertions holds:*

(i) *f satisfies the convex hull-like property in Ω.*

(ii) *There exists a non-empty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbf{R}^n$, there exists $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, the set $(g + \lambda f)(X)$ is supported at one of its points.*

Next, if $A \subseteq \mathbf{R}^n$ is a non-empty open set, $x \in A$ and $\varphi : A \to \mathbf{R}^n$ is a C^1 function, we denote by $\det(J_\varphi(x))$ the Jacobian determinant of φ at x.

Always in [2], as a joint application of Theorem 0.1 and the classical local inverse function theorem, we obtained the following result:

Theorem 0.2 ([2], Theorem 3.3)**.** *Let $f : \Omega \to \mathbf{R}^n$ be a C^1 function. Then, at least one of the following assertions holds:*

(a_1) *f satisfies the convex hull-like property in Ω.*

(a_2) *There exists a non-empty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbf{R}^n$ which is C^1 in X, there exists $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, the set*

$$\{x \in X : \det(J_{g+\lambda f}(x)) = 0\}$$

is non-empty.

The aim of the present addendum is to remark that, using Theorem 0.1 jointly with a very recent result by J. Saint Raymond [3], Theorem 0.2 can be improved in a very remarkable way. Actually, Saint Raymond proved the following very interesting result (for anything concerning topological dimension, we refer to [1]):

Theorem 0.3 ([3], Theorem 3.4)**.** *Let $A \subseteq \mathbf{R}^n$ be a non-empty open set and $\varphi : A \to \mathbf{R}^n$ a C^1 function such that the topological dimension of the set*

$$\{x \in A : \det(J_\varphi(x)) = 0\}$$

is not positive. Then, the function φ is open.

Finally, the improvement of Theorem 0.2, object of this addendum, is as follows:

Theorem 0.4. *Let $f : \Omega \to \mathbf{R}^n$ be a C^1 function. Then, at least one of the following assertions holds:*

(a_1) *f satisfies the convex hull-like property in Ω.*

(a_2) *There exists a non-empty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbf{R}^n$ which is C^1 in X, there exists $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, the topological dimension of the set*

$$\{x \in X : \det(J_{g+\lambda f}(x)) = 0\}$$

is greater than or equal 1.

Proof. Assume that (a_1) does not hold. Let X be an open set as in point (ii) of Theorem 0.1. Let $g : \Omega \to \mathbf{R}^n$ be a continuous function which is C^1 in X. Then, there is some $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, there exists $\hat{x} \in X$ such that the set $(g+\lambda f)(X)$ is supported at $g(\hat{x})+\lambda f(\hat{x})$. As already remarked, this implies that $g(\hat{x})+\lambda f(\hat{x}) \in \partial(g+\lambda f)(X)$ and so $(g+\lambda f)(X)$ is not open. Now, (a_2) is a direct consequence of Theorem 0.3. $\square$

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